Associating Finite Groups with *Dessins d'Enfants*

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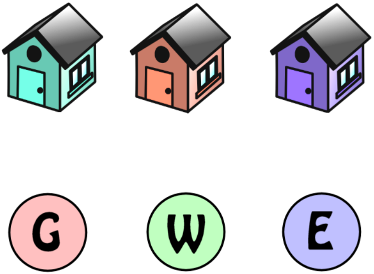
Outline of Talk

Utilities Problem

Question

Suppose there are three cottages, and each needs to be connected to the gas, water, and electric companies. Using a third dimension or sending any of the connections through another company or cottage are disallowed. Is there a way to make all nine connections without any of the lines crossing each other?



Graphs

- A (finite) graph is an ordered pair (V, E) consisting of vertices V and edges E.

$$\begin{aligned} v &= |V| = \text{the number of vertices} \\ e &= |E| = \text{the number of edges} \\ f &= |F| = \text{the number of faces} \end{aligned}$$

- A connected graph is a graph where, given any pair of vertices z_1 and z_2, one can traverse a path of edges from one to the other.

- Two vertices z_1 and z_2 are adjacent if there is an edge connecting them. A bipartite graph is a graph where the vertices V can be partitioned into two disjoint sets B and W such that no two edges $z_1,\ z_2 \in B$ (respectively, $z_1,\ z_2 \in W$) are adjacent.

- A planar graph is a graph that can be drawn such that the edges only intersect at the vertices.

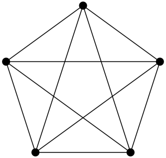
Which Graphs are Planar?

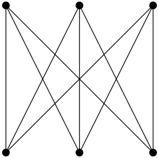
Theorem (Leonhard Euler, 1750)

A finite connected graph is planar if and only if $v - e + f = 2$.

Theorem (Kazimierz Kuratowski, 1930; Klaus Wagner, 1937)

A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.





Solution

Suppose there are three cottages, and each needs to be connected to the gas, water, and electric companies. There is no way to make all nine connections without any of the lines crossing each other because the Utility Graph has $v = 6$, $e = 9$, and $f = 3$.

Can We Draw Graphs on Other Objects?

Riemann Surfaces

A Riemann Surface is a triple $(X, \{U_\alpha\}, \{\mu_\alpha\})$ satisfying:

- Coordinate Charts and Maps: For some countable indexing set I,

$$X = \bigcup_{\alpha \in I} U_\alpha \qquad \text{and} \qquad \mu_\alpha : \ U_\alpha \hookrightarrow \mathbb{C}.$$

- Locally Euclidean: Each $\mu_\alpha(U_\alpha)$ is a connected, open subset of $\mathbb{C}$; and the composition $\mu_\beta \circ {\mu_\alpha}^{-1}$ is a smooth function.

$$\begin{array}{ccccc} \mathbb{C} & & X & & \mathbb{C} \\ \cup\mid & \xrightarrow{{\mu_\alpha}^{-1}} & \cup\mid & \xrightarrow{\mu_\beta} & \cup\mid \\ \mu_\alpha\left(U_\alpha \cap U_\beta\right) & & U_\alpha \cap U_\beta & & \mu_\beta\left(U_\alpha \cap U_\beta\right) \end{array}$$

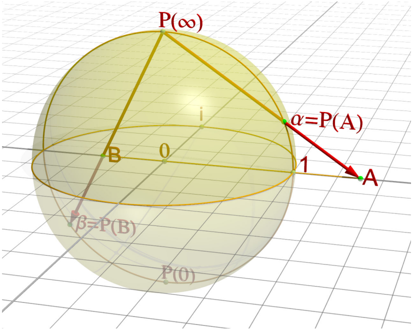
- Hausdorff: For distinct $z \in U_\alpha$ and $w \in U_\beta$ there exist open subsets

$$\begin{array}{c} \mu_\alpha(z) \in \mathcal{U}_\alpha \subseteq \mu_\alpha(U_\alpha) \\ \mu_\beta(w) \in \mathcal{U}_\beta \subseteq \mu_\beta(U_\beta) \end{array} \quad \text{such that} \quad {\mu_\alpha}^{-1}(\mathcal{U}_\alpha) \cap {\mu_\beta}^{-1}(\mathcal{U}_\beta) = \emptyset.$$

$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \simeq S^2(\mathbb{R})$ is an example. We always embed $X \hookrightarrow \mathbb{R}^3$.

The Sphere as The Extended Complex Plane

Through stereographic projection, we can establish a bijection between the unit sphere $S^2(\mathbb{R})$ and the extended complex plane $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.



Definition

Define stereographic projection as that map from the unit sphere to the complex plane.

$$\begin{array}{ccc} S^2(\mathbb{R}) & \xrightarrow{\sim} & \mathbb{P}^1(\mathbb{C}) \\ (u,\,v,\,w) = \left(\frac{2x}{x^2+y^2+1},\,\frac{2y}{x^2+y^2+1},\,\frac{x^2+y^2-1}{x^2+y^2+1}\right) & \mapsto & x + i\,y = \dfrac{u+i\,v}{1-w} \end{array}$$

Möbius Transformations

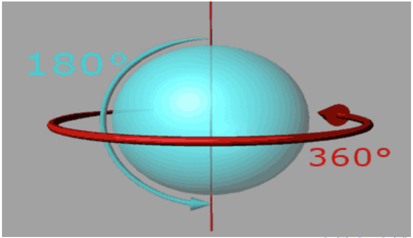
Definition

Rational functions of the form $f(z) = \dfrac{a\,z + b}{c\,z + d}$ where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C}) = \left\{ \gamma \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \;\middle|\; a\,d - b\,c \neq 0 \right\}$$

are called Möbius Transformations. We denote the collection of Möbius Transformations by $\mathrm{Aut}\big(\mathbb{P}^1(\mathbb{C})\big)$.

- For example, the function $f(z) = 1/z$ is also a Möbius transformation. Geometrically, this function represents a flip along the y-axis.



Finite Automorphism Groups of The Sphere

Proposition (Felix Klein)

The following groups are finite subgroups of $\mathrm{Aut}\big(\mathbb{P}^1(\mathbb{C})\big)$:

$$Z_n = \langle r | r^n = 1 \rangle : \quad r(z) = \zeta_n z$$

$$D_n = \langle r, s | s^2 = r^n = (sr)^2 = 1 \rangle : \quad r(z) = \zeta_n z \quad s(z) = \frac{1}{z}$$

$$A_4 = \langle r, s | s^2 = r^3 = (sr)^3 = 1 \rangle : \quad r(z) = \frac{z + 2\,\zeta_3}{z - \zeta_3} \quad s(z) = \frac{z+2}{z-1}$$

$$S_4 = \langle r, s | s^2 = r^3 = (sr)^4 = 1 \rangle : \quad r(z) = \frac{z + \zeta_4}{z - \zeta_4} \quad s(z) = \frac{z+1}{z-1}$$

$$A_5 = \langle r, s | s^2 = r^3 = (sr)^5 = 1 \rangle : \quad r(z) = \frac{\varphi - {\zeta_5}^3\, z}{\varphi\, {\zeta_5}^3\, z + 1} \quad s(z) = \frac{\varphi - z}{\varphi\, z + 1}$$

where $\zeta_n = e^{2\pi i/n}$ is a root of unity, and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Conversely, if G is a finite subgroup of $\mathrm{Aut}\big(\mathbb{P}^1(\mathbb{C})\big)$, then G is isomorphic to one of the five types of groups above.

Belyĭ's Theorem

Theorem (André Weil, 1956; Gennadiĭ Vladimirovich Belyĭ, 1979)

Let X be a compact, connected Riemann surface.

- *X is a smooth, irreducible, projective variety of dimension 1. In particular, X is an algebraic variety; that is, it can be defined by polynomial equations.*
- *If X can be defined by a polynomial equation $\sum_{i,j} a_{ij}\, z^i\, w^j = 0$ where the coefficients a_{ij} are not transcendental, then there exists a rational function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values.*
- *Conversely, if there exists rational function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values, then X can be defined by a polynomial equation $\sum_{i,j} a_{ij}\, z^i\, w^j = 0$ where the coefficients a_{ij} are not transcendental.*

Belyĭ Maps

Denote X as a Riemann Surface. We always embed $X \hookrightarrow \mathbb{R}^3$.

A rational function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is a map which is a ratio $\beta(z) = p(z)/q(z)$ in terms of relatively prime polynomials $p,\ q \in \mathbb{C}[X]$; define its degree as $\deg \beta = \max \left\{ \deg p,\ \deg q \right\}$.

Theorem (Fundamental Theorem of Algebra)

For $w \in \mathbb{P}^1(\mathbb{C})$, denote $\beta^{-1}(w) = \left\{ z \in X \,\middle|\, p(z) - w\, q(z) = 0 \right\}$. Then $\left|\beta^{-1}(w)\right| \leq \deg \beta$.

- $w \in \mathbb{P}^1(\mathbb{C})$ is said to be a critical value if $\left|\beta^{-1}(w)\right| \neq \deg \beta$.
- A Belyĭ map is a rational function β such that its collection of critical values w is contained within the set $\{0,\ 1,\ \infty\} \subseteq \mathbb{P}^1(\mathbb{C})$.

Examples

Denote $X = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \simeq S^2(\mathbb{R})$ as the Riemann Sphere.

$$\deg \beta = n : \qquad \beta(z) = z^n$$

$$\deg \beta = 2\,n : \qquad \beta(z) = \frac{4\,z^n}{\left(z^n + 1\right)^2}$$

$$\deg \beta = 12 : \qquad \beta(z) = \frac{\left(z^4 + 2\sqrt{2}\,z\right)^3}{\left(2\sqrt{2}\,z^3 - 1\right)^3}$$

$$\deg \beta = 24 : \qquad \beta(z) = \frac{1}{108}\,\frac{\left(z^8 + 14\,z^4 + 1\right)^3}{z^4\left(z^4 - 1\right)^4}$$

$$\deg \beta = 60 : \qquad \beta(z) = \frac{1}{1728}\,\frac{\left(z^{20} + 228\,z^{15} + 494\,z^{10} - 228\,z^5 + 1\right)^3}{z^5\left(z^{10} - 11\,z^5 - 1\right)^5}$$

Alexander Grothendieck (March 28, 1928 – ??)

http://en.wikipedia.org/wiki/Alexander_Grothendieck

Dessins d'Enfant

Fix a Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$. Denote the preimages

$$\begin{array}{cccc} B = \beta^{-1}(\{0\}) & W = \beta^{-1}(\{1\}) & E = \beta^{-1}([0,1]) & X \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C}) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \left\{ \begin{array}{c} \text{black} \\ \text{vertices} \end{array} \right\} & \left\{ \begin{array}{c} \text{white} \\ \text{vertices} \end{array} \right\} & \left\{ \text{edges} \right\} & \mathbb{R}^3 \end{array}$$

The bipartite graph $\Delta_\beta = (V, E)$ with vertices $V = B \cup W$ and edges E is called Dessin d'Enfant. We embed the graph on X in 3-dimensions.

> I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a *dessin* we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.
>
> – Alexander Grothendieck, *Esquisse d'un Programme* (1984)

Valency Lists and Passports

Definition

The valency of a vertex P_i is the number m_i of edges coming out of that vertex. The valency of a face R_k is the number t_k found by interchanging black vertices with midpoints of faces.

Definition

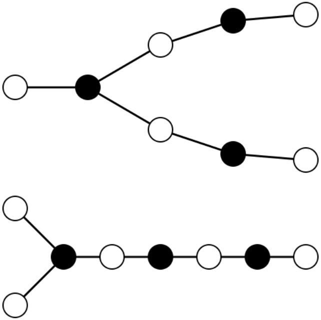
Say that we have a bipartite graph. A valency list is a collection of valencies $\{m_i \mid i \in I\}$, $\{n_j \mid j \in J\}$ and $\{t_k \mid k \in K\}$ for the "black" vertices, "white" vertices, and faces, respectively. If a_i, b_j, and c_k denotes the multiplicities of the numbers in these lists, we use the short-hand notation $\left[\prod_i {m_i}^{a_i},\ \prod_j {n_j}^{b_j},\ \prod_k {t_k}^{c_k}\right]$ as the passport.

The Degree Sum Formula is the identity

$$\begin{aligned} |I| + |J| + |K| - 2 &= \sum_{i \in I} m_i = \sum_{j \in J} n_j = \sum_{k \in K} t_k \\ &= \sum_i a_i\, m_i = \sum_j b_j\, n_j = \sum_k c_k\, t_k \end{aligned}$$

Example

- The valency lists {2, 2, 3}, {1, 1, 1, 2, 2} and {7} can be expressed as the passport $[2^2 \cdot 3,\ 1^3 \cdot 2^2,\ 7^1]$.
- This corresponds to two bipartite graphs:



- Valency lists do not uniquely determine a graph.

Riemann's Existence Theorem

We can always find a Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ that corresponds to a given passport.

Theorem

Say that we are given a bipartite graph on a compact, connected Riemann surface X, where the vertices and faces have valencies $\{m_i \mid i \in I\}$, $\{n_j \mid j \in J\}$ and $\{t_k \mid k \in K\}$. Then there exists a Belyi map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ such that

$$\beta^{-1}(0) = \{P_i \mid i \in I\}$$

$$\beta^{-1}(1) = \{Q_j \mid j \in J\}$$

$$\beta^{-1}(\infty) = \{R_k \mid k \in K\}$$

where the vertices P_i, Q_j, and R_k have valencies m_i, n_j, and t_k, respectively.

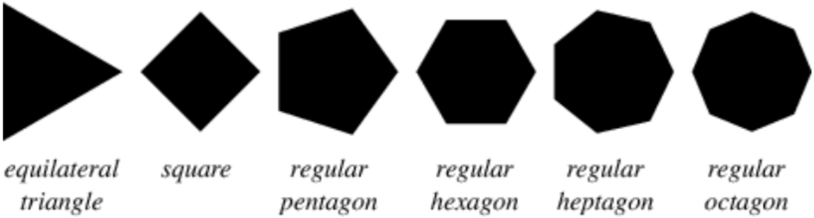
Which can be Realized as a Dessin d'Enfant?

Platonic, Archimedean, Catalan, and Johnson Solids

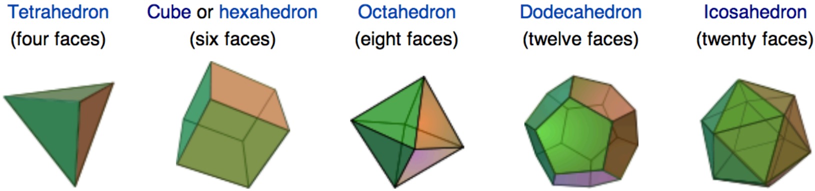
Definition

- A Platonic solid is a regular, convex polyhedron. They are named after Plato (424 BC – 348 BC). Aside from the regular polygons, there are five such solids.
- An Archimedean solid is a convex polyhedron that has a similar arrangement of nonintersecting regular convex polygons of two or more different types arranged in the same way about each vertex with all sides the same length. Discovered by Johannes Kepler (1571 – 1630) in 1620, they are named after Archimedes (287 BC – 212 BC). Aside from the prisms and antiprisms, there are thirteen such solids.
- A Catalan solid is a dual polyhedron to an Archimedean solid. They are named after Eugène Catalan (1814 – 1894) who discovered them in 1865. Aside from the bipyramids and trapezohedra, there are thirteen such solids.
- A Johnson solid is a convex polyhedron with regular polygons as faces but which is not Platonic or Archimedean. They are named after Norman Johnson (1930) who discovered them in 1966. There are ninety-two Johnson solids.

Platonic Solids



http://mathworld.wolfram.com/RegularPolygon.html



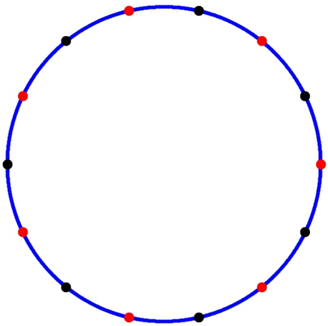
http://en.wikipedia.org/wiki/Platonic_solids

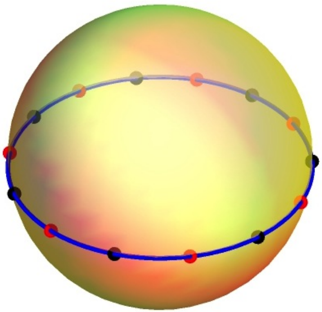
Rigid Rotations of the Platonic Solids

We have an action $\circ : PSL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.

- $Z_n = \langle r \mid r^n = 1 \rangle$ and $D_n = \langle r, s \mid s^2 = r^n = (s\,r)^2 = 1 \rangle$ are the rigid rotations of the regular convex polygons, with
$$r(z) = \zeta_n\, z \qquad \text{and} \qquad s(z) = \frac{1}{z}.$$
- $A_4 = \langle r, s \mid s^2 = r^3 = (s\,r)^3 = 1 \rangle$ are the rigid rotations of the tetrahedron, with
$$r(z) = \zeta_3\, z \qquad \text{and} \qquad s(z) = \frac{1-z}{2\,z+1}.$$
- $S_4 = \langle r, s \mid s^2 = r^3 = (s\,r)^4 = 1 \rangle$ are the rigid rotations of the octahedron and the cube, with
$$r(z) = \frac{\zeta_4 + z}{\zeta_4 - z} \qquad \text{and} \qquad s(z) = \frac{1-z}{1+z}.$$
- $A_5 = \langle r, s \mid s^2 = r^3 = (s\,r)^5 = 1 \rangle$ are the rigid rotations of the icosahedron and the dodecahedron, with
$$r(z) = \frac{(\zeta_5 + {\zeta_5}^4)\,\zeta_5 - z}{(\zeta_5 + {\zeta_5}^4)\,z + \zeta_5} \qquad \text{and} \qquad s(z) = \frac{(\zeta_5 + {\zeta_5}^4) - z}{(\zeta_5 + {\zeta_5}^4)\,z + 1}.$$

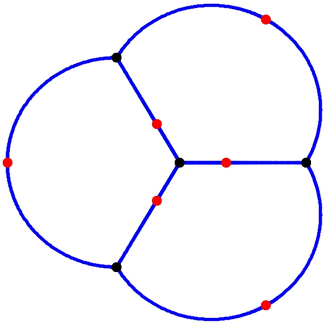
Rotation Group D_n: Regular Convex Polygon

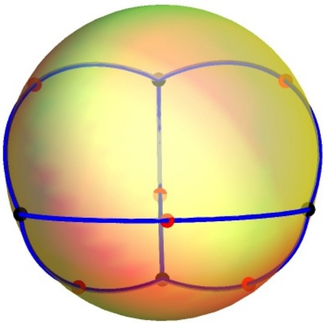




$$\beta(z) = \frac{(z^n + 1)^2}{4\,z^n} : \qquad v = n + n, \quad e = 2 \cdot n, \quad f = 2$$

Rotation Group A_4: Tetrahedron





$$\beta(z) = -\frac{64\, z^3 \left(z^3 - 1\right)^3}{\left(8\, z^3 + 1\right)^3} : \qquad v = 4 + 6, \quad e = 2 \cdot 6, \quad f = 4$$

Main Goals

- If we are given a valency list or a passport, how do we compute the corresponding Belyĭ map?
- How do we draw a Dessin if we have its passport?

Archimedean Solids and Catalan Solids

Proposition (Wushi Goldring, 2012)

Let $\varrho(w)$ be a rational function. The composition $\varrho \circ \beta$ is also a Belyĭ map for every Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ if and only if ϱ is a Belyĭ map which sends the set $\{0,\ 1,\ \infty\}$ to itself.

Proposition (Nicolas Magot and Alexander Zvonkin, 2000)

The following ϱ are Belyĭ maps which send the set $\{0,\ 1,\ \infty\}$ to itself.

$$\varrho(w) = \begin{cases} -(w-1)^2/(4\,w) & \text{is a rectification,} \\ (4\,w-1)^3/(27\,w) & \text{is a truncation,} \\ 1/w & \text{is a birectification,} \\ (4-w)^3/(27\,w^2) & \text{is a bitruncation,} \\ 4\,(w^2-w+1)^3/\left(27\,w^2\,(w-1)^2\right) & \text{is a rhombitruncation,} \\ (w+1)^4/\left(16\,w\,(w-1)^2\right) & \text{is a rhombification,} \\ \dfrac{7496192\,(w+\theta)^5}{25\,(3+8\,\theta)\,w\,\left(88\,w-(57\,\theta+64)\right)^3} & \text{is a snubification,} \end{cases}$$

Platonic Solid	Archimedean Solid	Catalan Solid	Rotation Group
Regular Polygon	Prism Antiprism	Bipyramid Trapezohedron	D_n
Tetrahedron	Truncated Tetrahedron	Triakis Tetrahedron	A_4
Octahedron Cube	Truncated Octahedron Truncated Cube Cuboctahedron Truncated Cuboctahedron Rhombicuboctahedron Snub Cube	Tetrakis Hexahedron Triakis Octahedron Rhombic Dodecahedron Disdyakis Dodecahedron Deltoidal Icositetrahedron Pentagonal Icositetrahedron	S_4
Icosahedron Dodecahedron	Truncated Icosahedron Truncated Dodecahedron Icosidodecahedron Truncated Icosidodecahedron Rhombicosidodecahedron Snub Dodecahedron	Pentakis Dodecahedron Triakis Icosahedron Rhombic Triacontahedron Disdyakis Triacontahedron Deltoidal Hexecontahedron Pentagonal Hexecontahedron	A_5

Johnson Solids

2013 PRiME's Results

Proposition (K. Biele, Y. Feng, D. Heras, and A. Tadde, 2013)

There are explicit Belyĭ maps β for

- Wheel/Pyramids (J_1, J_2)
- Cupola (J_3, J_4, J_5)
- Elongated Pyramids (J_7, J_8, J_9)
- Diminished Trapezohedron

which have rotation group $\mathrm{Aut}(\beta) \simeq Z_n$; and

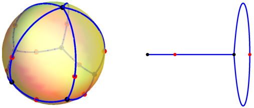
- Gyroelongated Bipyramid (J_{17})
- Dipole/Hosohedron
- Truncated Trapezohedron

which have rotation group $\mathrm{Aut}(\beta) \simeq D_n$.

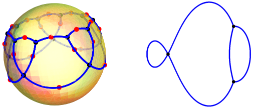
Approach

Following Magot and Zvonkin, reduce to easier cases using "hypermaps" $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, then composing $\beta = \phi \circ f$ where $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map as a function of either z^n or $4\,z^n/(z^n+1)^2$ such that $\mathrm{Aut}(f) \simeq Z_n$ or $\mathrm{Aut}(f) \simeq D_n$, respectively.

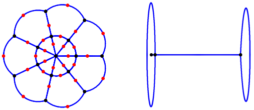
Results: Rotation Group Z_n



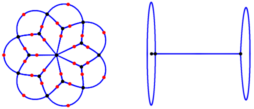
- Wheel/Pyramids (J_1, J_2)
- $\phi(w) = \dfrac{w^3\,(w+8)}{64\,(w-1)}$



- Cupola (J_3, J_4, J_5)
- $\phi(w) = \dfrac{4w^4(w^2-20w+105)^3}{(7w-48)^3(3w-32)^2(5w+12)}$

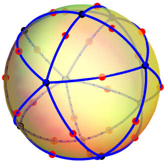
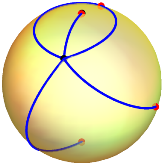


- Elongated Pyramids (J_7, J_8, J_9)
- $\phi(w) = \dfrac{4\,(835+872\sqrt{2})\,w^4\,(w-1)^3\,\left[(11+8\sqrt{2})\,w+1\right]}{\left[(8+9\sqrt{2})\,w+1\right]^3\,\left[(8-5\sqrt{2})\,w-1\right]}$

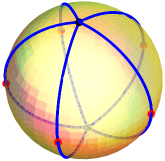


- Diminished Trapezohedron
- $\phi(w) = \dfrac{4\,(835-872\sqrt{2})\,w^4\,(w-1)^3\,\left[(11-8\sqrt{2})\,w+1\right]}{\left[(8-9\sqrt{2})\,w+1\right]^3\,\left[(8+5\sqrt{2})\,w-1\right]}$

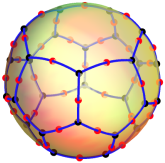
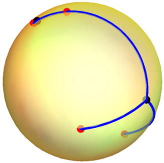
Results: Rotation Group D_n

- Gyroelongated Bipyramid (J_{17})
- $\phi(w) = \dfrac{1728\, w^5\, (w-1)}{25\,(11+18\varphi)\,(4w-\varphi^3)^3}$




- Dipole/Hosohedron
- $\phi(w) = w$

- Truncated Trapezohedron
- $\phi(w) = \dfrac{25\,(11+18\varphi)\, w^3 (\varphi^3\, w - 4)^3}{1728(w-1)}$

2013 PRIME Limitations

- Rudimentary software was developed to help with the research
- Only a few Johnson Solids were really examined for Dessin d'Enfants

2014 PRiME's Approach

Definition

A Johnson solid is a convex polyhedron with regular polygons as faces but which is not Platonic or Archimedean. They are named after Norman Johnson (1930) who discovered them in 1966. There are ninety-two Johnson solids.

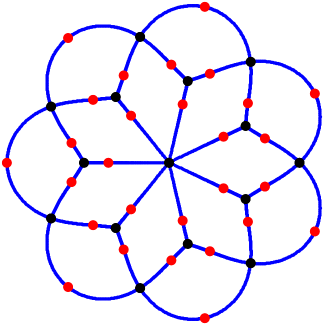
Proposition

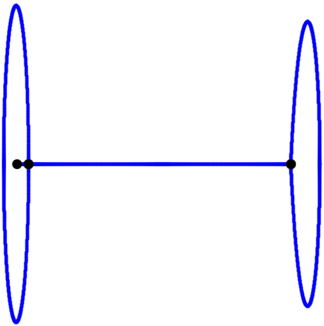
The symmetry groups of the Johnson solids are either cyclic Z_n or dihedral D_n.

Approach

Following Magot and Zvonkin, reduce to easier cases using "hypermaps" $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, then composing $\beta = \phi \circ f$ where $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map as a function of either z^n or $4\, z^n/(z^n + 1)^2$ such that $\mathrm{Aut}(f) \simeq Z_n$ or $\mathrm{Aut}(f) \simeq D_n$, respectively.

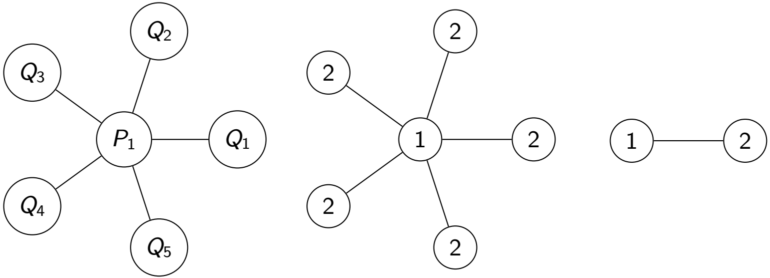
Reduction to Hypergraphs





- Diminished Trapezohedron
- $\phi(w) = \dfrac{4\,(835 - 872\sqrt{2})\,w^4\,(w-1)^3\,\left[(11 - 8\sqrt{2})\,w + 1\right]}{\left[(8 - 9\sqrt{2})\,w + 1\right]^3\,\left[(8 + 5\sqrt{2})\,w - 1\right]}$

Reduction by Hypergraphs



2014 PRiME's Results

Proposition (E. Baeza, L. Baeza, C. Lawrence, and C. Wang, 2014)

There are explicit Belyĭ maps β for

- Wheel/Pyramids (J_1, J_2)
- Cupola (J_3, J_4, J_5)
- Rotunda (J_6)
- Elongated Pyramids (J_7, J_8, J_9)
- Diminished Trapezohedron
- Gyroelongated Pyramid (J_{10}, J_{11})
- Augmented Triangular Prism (J_{49})
- Tridiminished Icosahedron (J_{63})
- Augmented Tridiminished Icosahedron (J_{64})

which have rotation group $\mathrm{Aut}(\beta) \simeq Z_n$.

2014 PRiME's Results (cont'd)

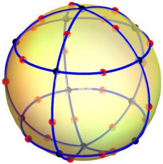
Proposition (E. Baeza, L. Baeza, C. Lawrence, and C. Wang, 2014)

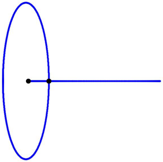
There are explicit Belyĭ maps β for

- Bipyramid (J_{12}, J_{13})
- Elongated Bipyramid (J_{14}, J_{15}, J_{16})
- Gyroelongated Bipyramid (J_{17})
- Gyrobifastigium (J_{26})
- Orthobicupola (J_{27}, J_{28}, J_{30})
- Gyrobicupola (J_{29}, J_{31})
- Elongated Orthobicupola (J_{35}, J_{38})
- Elongated Gyrobicupola (J_{36}, J_{37}, J_{39})
- Dipole/Hosohedron
- Truncated Trapezohedron
- Bifrustum/Truncated Bipyramid
- Triaugmented Prism (J_{51})
- Parabiaugmented Prism (J_{55})
- Triaugmented Hexagonal Prism (J_{57})
- Parabiaugmented Dodecahedron (J_{59})
- Snub Disphenoid (J_{84})
- Snub Square Antiprism (J_{85})
- Bilunabirotunda (J_{91})

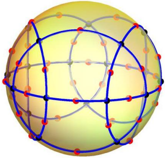
which have rotation group $\text{Aut}(\beta) \simeq D_n$.

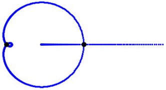
Results: Rotation Group D_n



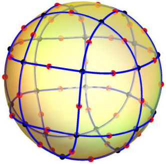


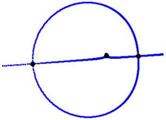
- Elongated Bipyramid (J_{14}, J_{15}, J_{16})
- $\phi(w) = \dfrac{w\,(32\,w - 5)^4}{(80\,w + 1)^3}$





- Orthobicupola (J_{27}, J_{28}, J_{30})
- $\phi(w) = \dfrac{(w - 2.411)^4\,(w + 0.138)^4}{w\,(w + 3.086)^3\,(w - 0.441)^4}$





- Elongated Gyrobicupola (J_{36}, J_{37}, J_{39})
- $\phi(w) = \dfrac{\left[(w^3+(0.739+0.223\,i)w^2-(0.754+0.034\,i)\,w+(0.002-0.020\,i)\right]^4}{w\,\left[w-(0.041-0.283\,i)\right]^3\,\left[w^2-(2.004+0.189\,i)\,w+(0.005+0.234\,i)\right]^4}$

How do we Compute these Hypermaps?

Computing Belyĭ Maps

Say that $\Delta = (B \cup W, E)$ is a loopless, connected, bipartite planar graph on the Riemann Sphere $S^2(\mathbb{R}) \simeq \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

- The "black" vertices are $B = \{P_i \mid i \in I\}$ for some set I, where each "black" vertex P_i has m_i edges incident.
- The "white" vertices are $W = \{Q_j \mid j \in J\}$ for some set J, where each "white" vertex Q_j has n_j edges incident, i.e., "black" vertices adjacent.
- The midpoints of the faces are $F = \{R_k \mid k \in K\}$ for some set K, where each face R_k has t_k "white" vertices adjacent.

The finite sets $\{m_i \mid i \in I\}$, $\{n_j \mid j \in J\}$, and $\{t_k \mid k \in K\}$ are called the valencies of the graph.

Computing Belyĭ Maps

Problem

Find a Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that its Dessin d'Enfant is Γ. That is, find two relatively prime polynomials $p(z)$ and $q(z)$ such that

- $B = \beta^{-1}(0) = \{P_i \in \mathbb{P}^1(\mathbb{C}) \,|\, p(P_i) = 0\}$.
- $W = \beta^{-1}(1) = \{Q_j \in \mathbb{P}^1(\mathbb{C}) \,|\, p(Q_j) = q(Q_j)\}$.
- $F = \beta^{-1}(\infty) = \{R_k \in \mathbb{P}^1(\mathbb{C}) \,|\, q(R_k) = 0\}$.

We must have the factorizations

$$\begin{aligned} p(z) &= +p_0 \prod_{i \in I} \left[z - P_i \right]^{m_i} \\ p(z) - q(z) &= = -q_0 \prod_{j \in J} \left[z - Q_j \right]^{n_j} \\ q(z) &= -r_0 \prod_{k \in K} \left[z - R_k \right]^{t_k} \end{aligned} \implies \beta(z) = -\frac{p_0}{r_0} \frac{\prod_{i \in I} \left[z - P_i \right]^{m_i}}{\prod_{k \in K} \left[z - R_k \right]^{t_k}}$$

for some nonzero constants p_0, q_0, and r_0.

Computing Belyĭ Maps

Existence Theorem (Bernhard Riemann, 1850's)

At least one such solution exists!

Approach

Given valencies $\{m_i \mid i \in I\}$, $\{n_j \mid j \in J\}$, and $\{t_k \mid k \in K\}$ such that

$$|I| + |J| + |K| - 2 = \sum_{i \in I} m_i = \sum_{j \in J} n_j = \sum_{k \in K} t_k,$$

find nontrivial P_i, Q_j, R_k, p_0, q_0, $r_0 \in \mathbb{P}^1(\mathbb{C})$ such that

$$p_0 \prod_{i \in I} \Big[z - P_i \Big]^{m_i} + q_0 \prod_{j \in J} \Big[z - Q_j \Big]^{n_j} + r_0 \prod_{k \in K} \Big[z - R_k \Big]^{t_k} = 0$$

identically as a polynomial in z.

Future Work: Rotation Group Z_n

- Elongated Cupola (J_{18}, J_{19}, J_{20})
- Elongated Rotunda (J_{21})
- Gyroelongated Cupola (J_{22}, J_{23}, J_{24})
- Gyroelongated Rotunda (J_{25})
- Orthocupolarotunda (J_{32})
- Gyrocupolarotunda (J_{33})
- Elongated Orthocupolarotunda (J_{40})
- Elongated Gyrocupolarotunda (J_{41})
- Augmented Prism (J_{49}, J_{52}, J_{54})
- Biaugmented Prism (J_{50}, J_{53})
- Metabiaugmented Prism (J_{56})
- Augmented Dodecahedron (J_{58})
- Metabiaugmented Dodecaherdon (J_{60})
- Triaugmented Dodecahedron (J_{61})
- Metabidiminished Icosahedron (J_{62})
- Augmented Truncated Tetrahedron (J_{65})
- Augmented Truncated Cube (J_{66})
- Augmented Truncated Dodecahedron (J_{68})
- Metabiaugmented Truncated Dodecahedron (J_{70})
- Triaugmented Truncated Dodecahedron (J_{71})
- Gyrate Rhombicosidodecahedron (J_{72})
- Metabigyrate Rhombicosidodecahedron (J_{74})
- Trigyrate Rhombicosidodecahedron (J_{75})
- Diminished Rhombicosidodecahedron (J_{76})
- Paragyrate Diminished Rhombicosidodecahedron (J_{77})
- Metabidiminished Rhombicosidodecahedron (J_{81})
- Tridiminished Rhombicosidodecahedron (J_{83})
- Sphenocorona (J_{86})
- Sphenomegacorona (J_{88})
- Hebesphenomegacorona (J_{89})
- Triangular Hebesphenorotunda (J_{92})

Future Work: Rotation Group D_n

- Orthobirotunda (J_{34})
- Elongated Orthobirotunda (J_{42})
- Elongated Gyrobirotunda (J_{43})
- Gyroelongated Bicupola (J_{44}, J_{45}, J_{46})
- Gyroelongated Birotunda (J_{48})
- Biaugmented Truncated Cube (J_{67})
- Parabiaugmented Truncated Dodecahedron(J_{69})
- Parabigyrate Rhombicosidodecahedron (J_{73})
- Parabidiminished Rhombicosidodecahedron (J_{80})
- Disphenocingulum (J_{90})

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Questions?